

MIDSEMESTRAL

Elementary Number Theory

Instructor: Ramdin Mawia

Marks: 30

Time: September 10, 2026; 14:00–16:00.

Attempt THREE problems. The maximum you can score is 30.

1. Let p be an odd prime and $r \in \mathbb{Z}$ be a primitive root mod p . Prove or disprove (**any two**): **11**

i. $r^{(p-1)/2} \equiv -1 \pmod{p}$.

ii. If $p \equiv 1 \pmod{4}$, then $-r$ is also a primitive root mod p .

iii. If $r_1 \not\equiv r \pmod{p}$ is another primitive root mod p , then rr_1 is never a primitive root mod p .

2. Prove or disprove: A positive integer n is expressible as a sum of consecutive nonnegative integers if and only if it is not a power of 2. **11**

3. How many recurring digits are there in the decimal expansion of $1/337$, given that 337 is a prime? **11**

4. Let $p \geq 5$ be a prime. Write **11**

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p} = \frac{a}{pb},$$

where a and b are integers. Prove that $a \equiv b \pmod{p^3}$.

5. Prove that there are infinitely many primes of the form $4k + 1$. [*Hint.* Look at the prime factors of $N := (n!)^2 + 1$ for any integer $n \geq 2$.] **11**

6. Let p be an odd prime. When do we say that an integer r is a *primitive root* for p ? Let $(r, p) = 1$ and $p - 1 = p_1^{e_1} \cdots p_k^{e_k}$ be the prime factorisation of $p - 1$. Prove that r is a primitive root for p if and only if **11**

i. $r^{p-1} \equiv 1 \pmod{p}$;

ii. $r^{(p-1)/p_i} \not\equiv 1 \pmod{p}$ for any $i = 1, \dots, k$.

7. Find the smallest positive integer x satisfying the simultaneous congruences **11**

$$x \equiv 2 \pmod{5},$$

$$x \equiv 3 \pmod{7},$$

$$x \equiv 5 \pmod{11}.$$

–The End–